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THE USE OF A BEAM SPACE REPRESENTATION AND NONLINEAR PROGRAMMING IN PHASE-ONLY NULLING

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A new computational method is presented for calculating the phases required for minimized weight perturbation, phase-only null synthesis in linear array antenna patterns. The method uses nonlinear programming algorithms to determine the coefficients of a beam space representation of the optimal phase perturbations. The derivatives of the objective and null constraint functions with respect to the beam coefficients, required by the nonlinear programming algorithms, are calculated from analytic expressions. The

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20. Abstract (Contd)

beam space representation allows the number of unknowns to be reduced from half the number of array elements to the number of imposed null locations, and results in a significant reduction in computation time from that required to calculate the phase perturbations directly.

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I. INTRODUCTION

The considerable recent interest in the use of phase-only control of the element weights of array antennas for adaptive nulling[1-9] reflects the growth in importance of both phased arrays and adaptive processing. Phase-only null synthesis in array antenna patterns is also of current interest[10-18] because it can help establish limits to what can be achieved adaptively, and possibly suggest adaptive procedures.

Phase-only null synthesis presents analytic and computational difficulties not present when both amplitude and phase of the element weights can be freely perturbed. The restriction of the weight perturbations to be of the phases only, makes the nulling problem nonlinear and not solvable analytically. Approximations and/or numerical techniques must be used to calculate the phases required to impose nulls in the pattern. As with nulling with combined phase and amplitude control, the number of imposed nulls is typically considerably less than the available number of degrees of freedom (one half the number of elements for phase-only nulling in real antenna patterns) and so additional conditions must be imposed to define a unique solution. Since in either null synthesis or adaptive nulling it is

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Because of the large number of references cited above, they will not be listed here. See References, page 29.

generally desired to preserve the original pattern intact as much as possible apart from the immediate vicinity of the imposed null locations, a natural condition to be added to the requirement of nulls at specified locations is that the perturbation of the pattern be minimized in a mean square sense. For a linear array with half-wavelength interelement spacing this is equivalent to requiring that the sum of the squared absolute values of the weight perturbations be a minimum. A more general condition requiring minimization of a weighted sum of the squared absolute weight perturbations can provide additional flexibility in nulling that may be desirable in certain situations.

In a series of recent reports we have investigated various approaches to calculating the phases for minimized weight perturbation, phase-only null synthesis. A straightforward linear approximation method, reasonably effective for small phase perturbations, is discussed in Reference 13; a method based on iterated linearization of the phase-only nulling equations is described in Reference 14; a method consisting of obtaining the best phase-only approximation to the combined phase and amplitude perturbation solution of the nulling problem is reported in Reference 17; and the use of nonlinear programming techniques to calculate the phase perturbations is the subject of Reference 18. Of these methods the nonlinear programming approach is the most general and effective. In this report we describe a variation of the nonlinear programming method based on the work of Baird and Rassweiler.

In their basic paper on phase-only nulling,[1] Baird and Rassweiler considered the problem of minimizing the mean square difference between a desired signal and the output of a linear array by varying the phases only of the array elements. The amplitudes of the element weights were assumed equal to unity. The sources of the signal and the interferences were modeled as discrete, single-frequency directional emitters. The vector of optimal phases was shown to be expressible as the phase of a linear combination of the complex conjugates of the vectors giving the signal and interferences as received at the elements of the array. This representation was referred to as the "beam space decomposition," and the coefficients of the vectors as "beam space coefficients" or "beam coefficients."

The purpose of this report is to apply the analytic method of Baird and Rassweiler to the problem of imposing nulls in the pattern of a linear array of elements by varying the phases only of the element weights, subject to the condition that the perturbations of the weights be minimized in a weighted least-squares sense. In Section 2 we obtain a representation of the desired phases similar to that of Baird and Rassweiler, but slightly more general in that the solution allows for an arbitrary amplitude taper of the element weights, and for any choice of the coefficients entering into the weighted sum of the squared weight perturbations. When the phase perturbations are small the perturbed pattern can be interpreted as the original pattern plus a sum of cancellation beams directed towards the imposed null locations and the locations symmetric with respect to the mainbeam. The major contribution of this report to the literature on phase-only nulling is contained in Section 3, in which it is shown that a highly efficient method of calculating the phase perturbations for minimized weight perturbation null synthesis is to use nonlinear programming computer algorithms to calculate the coefficients in the beam space representation of the phase perturbations. The use of the beam space representation enables the number of unknowns to be reduced from half the number of elements to the number of imposed null locations. Section 4 of the report contains a brief discussion of the use of the beam space representation in adaptive phase-only nulling.

2. ANALYSIS

We consider a linear array of N equispaced isotropic elements (Figure 1) whose field pattern is given by

$$p_o(u) = \sum_{n=1}^{N} a_n e^{jd_n u} \quad . \tag{1}$$

In Eq. (1) the $\{a_n\}$ are the complex element weights,

$$d_n = \frac{N-1}{2} - (n-1) = -d_{N-n+1} \quad , \quad n = 1, 2, \ldots, N \quad , \tag{2}$$

and

$$u = \frac{2\pi}{\lambda} d \sin\theta \quad ,$$

where

λ = wavelength,

d = interelement spacing, and

θ = angle measured from broadside to the array.

The phase reference is taken to be the center of the array. Let ϕ_n, n = 1, 2, ..., N, be the set of perturbations that (a) imposes nulls in the pattern at the locations $u = u_k$, k = 1, 2, ..., K:

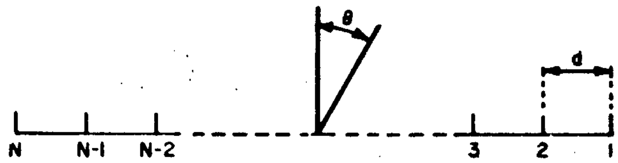

Figure 1. Geometry of Array

$$\sum_{n=1}^{N} a_n e^{j\phi_n} e^{jd_n u_k} = 0 \quad , \quad k = 1, 2, \ldots, K \ . \tag{3}$$

and (b) minimizes the weighted sum of the squares of the absolute values of the element weight perturbations

$$F = \sum_{n=1}^{N} c_n \left| a_n \left(e^{j\phi_n} - 1 \right) \right|^2 \ . \tag{4}$$

The weighting coefficients $\{c_n\}$ in Eq. (4) are assumed real and positive.*

The method we employ to derive a beam space representation for the phase perturbations is that of Baird a 1 Rassweiler.[1] We let

$$w_n = e^{j\phi_n} \quad , \quad n = 1, 2, \ldots, N \ , \tag{5}$$

and make the null constraints purely real by multiplying the left-hand side of Eq. (3) by its complex conjugate. The null constraints then become

$$\sum_{m=1}^{N} \sum_{n=1}^{N} a_m^* a_n w_m^* w_n e^{-j(d_m - d_n)u_k} = 0 \quad , \quad k = 1, 2, \ldots, K \ . \tag{6}$$

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*The choice $c_n = 1$ for all n makes F the sum of the squares of the absolute values of the weight perturbations. For half-wavelength spacing of the array elements, this is equivalent to minimizing the mean square pattern perturbation for θ from $-\pi/2$ to $\pi/2$. Other choices of the c_n with practical application to null synthesis are also possible;[13, 15] for example, $c_n = 1/|a_n|^2$.

while

$$F = \sum_{n=1}^{N} c_n |a_n|^2 |w_n - 1|^2$$

$$= \sum_{n=1}^{N} c_n |a_n|^2 (2 - 2\,\mathrm{Re}[w_n]) \quad . \tag{7}$$

Since the $\{\phi_n\}$ are assumed to be real phase perturbations, the $\{w_n\}$ by Eq. (5) have unit modulus. This is expressed by an additional set of N constraints

$$w_n^* w_n = 1 \quad , \quad n = 1, 2, \ldots, N \quad . \tag{8}$$

We now form the Lagrangian

$$L = F + \sum_{k=1}^{K} \mu_k C_k + \sum_{n=1}^{N} \lambda_n D_n \quad .$$

where from Eqs. (6) and (8) respectively,

$$C_k = \sum_{m=1}^{N} \sum_{n=1}^{N} a_m^* a_n e^{-j(d_m - d_n)u_k} w_m^* w_n \quad , \quad k = 1, 2, \ldots, K \, ,$$

and

$$D_n = w_n^* w_n - 1 \quad , \quad n = 1, 2, \ldots, N \quad . \tag{9}$$

The $\{\mu_k\}$ and $\{\lambda_n\}$ are (real) Lagrangian multipliers. A necessary condition for F to have a stationary point is that the gradient* of the Lagrangian with respect to $\underline{w} = [w_1, w_2, \ldots, w_N]^T$ be zero:[19]

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*The gradient with respect to a complex vector $\underline{z}$ is defined[20] to be $\nabla_{\underline{z}} = \nabla_{\mathrm{Re}[\underline{z}]} + j\nabla_{\mathrm{Im}[\underline{z}]}$.

19. Fletcher, R. (1981) Practical Methods of Optimization; Vol. 2, Constrained Optimization, John Wiley & Sons, New York, Ch. 9.

20. Morse, P.M., and Feshback, H. (1953) Methods of Theoretical Physics, Part I, McGraw-Hill, N.Y., p. 351.

$$\nabla_{\underline{w}} L = \nabla_{\underline{w}} F + \nabla_{\underline{w}} \sum_{k=1}^{K} \mu_k C_k + \nabla_{\underline{w}} \sum_{n=1}^{K} \lambda_n D_n = 0 \quad . \tag{10}$$

Now from Eq. (7)

$$\begin{aligned} \nabla_{\underline{w}} F &= -2 \nabla_{\underline{w}} \mathrm{Re}(\underline{w}^T \underline{\alpha}) \\ &= -2\underline{\alpha} \end{aligned} \tag{11}$$

where we have let $\underline{\alpha}$ denote the vector

$$\underline{\alpha} = \left[c_1 |a_1|^2 , c_2 |a_2|^2 , \ldots , c_N |a_N|^2 \right]^T \quad . \tag{12}$$

The null constraint functions, $\{C_k\}$, are Hermitian quadratic forms and can be written as

$$\begin{aligned} C_k &= \underline{w}^\dagger \underline{R}_k \underline{w} \\ &= \underline{w}^\dagger \underline{v}_k \underline{v}_k^\dagger \underline{w} \quad , \end{aligned}$$

where

$$\underline{v}_k = \left[a_1^* e^{-jd_1 u_k} , \ldots , a_N^* e^{-jd_N u_k} \right]^T \quad , \quad k = 1, 2, \ldots, K \quad . \tag{13}$$

The elements of the Hermitian matrix R_k are

$$[\underline{R}_k]_{mn} = [\underline{R}_k]^*_{nm} = a_m^* a_n e^{-j(d_m - d_n) u_k} \quad , \quad m, n = 1, 2, \ldots, N \quad .$$

Letting $\underline{R}$ be the Hermitian matrix

$$\begin{aligned} \underline{R} &= \sum_{k=1}^{K} \mu_k \underline{R}_k \\ &= \sum_{k=1}^{K} \mu_k \underline{v}_k \underline{v}_k^\dagger \quad . \end{aligned}$$

then gives

$$\nabla_{\underline{w}} \sum_{k=1}^{K} \mu_k C_k = \nabla_{\underline{w}} \underline{w}^{\dagger} \underline{R}\,\underline{w}$$

$$= 2\,\underline{R}\,\underline{w} \tag{14}$$

by using the formula[21] for the complex gradient of a Hermitian quadratic form. Finally,

$$\nabla_{\underline{w}} \sum_{n=1}^{N} \lambda_n D_n = 2\{\lambda_1 w_1, \ldots, \lambda_N w_N\}$$

$$= 2\,\underline{\Lambda}\,\underline{w} \quad , \tag{15}$$

where

$$\underline{\Lambda} = \begin{bmatrix} \lambda_1 & & \emptyset \\ & \lambda_2 & \\ & & \ddots & \\ \emptyset & & & \lambda_N \end{bmatrix} . \tag{16}$$

Substituting Eqs. (11), (14), and (15), in Eq. (10) we obtain

$$-2\underline{\alpha} + 2\,\underline{R}\,\underline{w} + 2\,\underline{\Lambda}\,\underline{w} = 0$$

or

$$(\underline{R} + \underline{\Lambda})\,\underline{w} = \underline{\alpha} \quad ,$$

and hence

$$\underline{w} = (\underline{R} + \underline{\Lambda})^{-1}\,\underline{\alpha} \quad . \tag{17}$$

Equations (6), (8), and (17) form a system of 2N + K equations for the 2N + K unknowns $\{w_n\}$, $\{\lambda_n\}$, and $\{\mu_k\}$. Since the constraints, Eq. (8), specifying the use of phase-only weight control are nonlinear, the system of equations can be solved

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21. Hudson, J. E. (1981) Adaptive Array Principles, Institution of Electrical Engineers, London, p. 245.

only by the use of numerical techniques, such as nonlinear programming. However, Eq. (17) can be used to derive a convenient representation of the phase perturbations. Following Baird and Rassweiler we define a sequence of matrices $\underline{B}_m$ by

$$\underline{B}_m = \underline{A} + \sum_{k=1}^{m} \mu_k \underline{v}_k \underline{v}_k^\dagger \quad , \quad m = 1, 2, \ldots, K \ ,$$

$$\underline{B}_o = \underline{A} \ .$$

with the vectors $\underline{v}_k$ defined by Eq. (13). Using the matrix inversion formula,[22]

$$(\underline{A} + \underline{z}\,\underline{z}^\dagger)^{-1} = \underline{A}^{-1} - \frac{\underline{A}^{-1}\underline{z}\,\underline{z}^\dagger \underline{A}^{-1}}{1 + \underline{z}^\dagger \underline{A}^{-1} \underline{z}} \ ,$$

we see that the $\underline{B}_m$ satisfy the backwards recursion relation

$$\underline{B}_m^{-1} = \underline{B}_{m-1}^{-1} - \frac{\mu_m \underline{B}_{m-1}^{-1} \underline{v}_m \underline{v}_m^\dagger \underline{B}_{m-1}^{-1}}{1 + \mu_m \underline{v}_m^\dagger \underline{B}_{m-1}^{-1} \underline{v}_m} \tag{18}$$

$$\underline{B}_o^{-1} = \underline{A}^{-1} \ . \tag{19}$$

Since

$$\underline{B}_K = \underline{A} + \underline{R} \ ,$$

from Eq. (17)

$$\underline{w} = \underline{B}_K^{-1} \underline{\alpha}$$

$$= \left[\underline{B}_{K-1}^{-1} - \frac{\mu_K \underline{B}_{K-1}^{-1} \underline{v}_K \underline{v}_K^\dagger \underline{B}_{K-1}^{-1}}{1 + \mu_k \underline{v}_K^\dagger \underline{B}_{K-1}^{-1} \underline{v}_K} \right] \underline{\alpha}$$

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22. Householder, A.S. (1964) The Theory of Matrices in Numerical Analysis, Blaisdell, N.Y., p. 123.

$$= \underline{B}_{K-1}^{-1}(\underline{\alpha} - \gamma_K \underline{v}_K) \quad .$$

where γ_K is the scalar

$$\gamma_K = \frac{\mu_K \underline{v}_K^{\dagger} \underline{B}_{K-1}^{-1} \underline{\alpha}}{1 + \mu_K \underline{v}_K^{\dagger} \underline{B}_{K-1}^{-1} \underline{v}_K} \quad .$$

We now repeat the process, using Eq. (18) and letting

$$\underline{t}_{K-1} = \underline{\alpha} - \gamma_K \underline{v}_K \quad ,$$

so that

$$\underline{w} = \underline{B}_{K-1}^{-1} \underline{t}_{K-1}$$

$$= \left[\underline{B}_{K-2}^{-1} - \frac{\mu_{K-1} \underline{B}_{K-2}^{-1} \underline{v}_{K-1} \underline{v}_{K-1}^{\dagger} \underline{B}_{K-2}^{-1}}{1 + \mu_{K-1} \underline{v}_{K-1}^{\dagger} \underline{B}_{K-2}^{-1} \underline{v}_{K-1}} \right] \underline{t}_{K-1}$$

$$= \underline{B}_{K-2}^{-1}(\underline{t}_{K-1} - \gamma_{K-1} \underline{v}_{K-1})$$

$$= \underline{B}_{K-2}^{-1}(\underline{\alpha} - \gamma_K \underline{v}_K - \gamma_{K-1} \underline{v}_{K-1}) \quad ,$$

where

$$\gamma_{K-1} = \frac{\mu_{K-1} \underline{v}_{K-1}^{\dagger} \underline{B}_{K-2} \underline{t}_{K-1}}{1 + \mu_{K-1} \underline{v}_{K-1}^{\dagger} \underline{B}_{K-2}^{-1} \underline{v}_{K-1}} \quad .$$

Continuing the process, and using Eq. (19) at the end, we obtain

$$\underline{w} = \underline{A}^{-1}\left(\underline{\alpha} - \sum_{k=1}^{K} \gamma_k \underline{v}_k\right) \quad . \tag{20}$$

where

$$\gamma_k = \frac{\mu_k \underline{v}_k^\dagger \underline{B}_{k-1}^{-1} \underline{t}_k}{1 + \mu_k \underline{v}_k^\dagger \underline{B}_{k-1}^{-1} \underline{v}_k} \quad , \quad k = 1, 2, \ldots, K \ ,$$

and

$$\underline{t}_k = \begin{cases} \underline{\alpha} - \sum_{m=k+1}^{K} \gamma_k \underline{v}_k & , \quad k = 1, 2, \ldots, K-1 \ , \\ \underline{\alpha} & , \quad k = K \ . \end{cases}$$

In component form, Eq. (20) is, referring to Eqs. (5), (16), (12), and (13),

$$e^{j\phi_n} = \frac{1}{\lambda_n} \left(c_n |a_n|^2 - \sum_{k=1}^{K} \gamma_k a_n^* e^{-jd_n u_k} \right) \quad , \quad n = 1, 2, \ldots, N \ .$$

Expressing the unperturbed element weights $\{a_n\}$ in the magnitude-and-phase form $a_n = |a_n| \exp(j\psi_n)$,

$$e^{j\phi_n} = \frac{1}{\lambda_n} \left(c_n |a_n|^2 - |a_n| \, e^{-j\psi_n} \sum_{k=1}^{K} \gamma_k e^{-jd_n u_k} \right)$$

$$= \frac{|a_n|}{\lambda_n} \left(c_n |a_n| - e^{-j\psi_n} \sum_{k=1}^{K} \gamma_k e^{-jd_n u_k} \right) \quad , \quad n = 1, 2, \ldots, N \ , \qquad (21)$$

and the total phase dependence of the element weights is then

$$e^{j(\phi_n + \psi_n)} = \frac{|a_n|}{\lambda_n} \left(c_n a_n - \sum_{k=1}^{K} \gamma_k e^{-jd_n u_k} \right) \quad , \quad n = 1, 2, \ldots, N \ . \qquad (22)$$

Assuming that the Lagrangian multipliers $\{\lambda_n\}$ of the constraint functions $\{D_n\}$ of Eq. (9) are all positive, the factor $|a_n|/\lambda_n$ in Eqs. (21) and (22) has no effect on the phases. Hence, the phase perturbations are given by

$$\phi_n = \text{phase}\left(c_n|a_n| - e^{-j\psi_n}\sum_{k=1}^{K}\gamma_k e^{-jd_n u_k}\right) \quad , \quad n = 1, 2, \ldots, N \; . \tag{23}$$

and the total phases of the element weights by

$$\phi_n + \psi_n = \text{phase}\left(c_n a_n - \sum_{k=1}^{K}\gamma_k e^{-jd_n u_k}\right) \quad , \quad n = 1, 2, \ldots, N \; . \tag{24}$$

For the special case that $|a_n| = c_n = 1$, Eq. (24) reduces to the form of Eq. (9) of Baird and Rassweiler (see Appendix B).

The coefficients $\{\gamma_k\}$ in Eqs. (23) and (24) are in general complex. However in the important special case that (a) the unperturbed pattern is real, and hence that the unperturbed element weights are conjugate symmetric,

$$a_{N-n+1} = a_n^* \quad , \quad n = 1, 2, \ldots, N \; ;$$

and (b) the weighting coefficients $\{c_n\}$ are chosen to be even symmetric,

$$c_{N-n+1} = c_n \quad , \quad n = 1, 2, \ldots, N \; ;$$

it has been shown[23] that the phase perturbations are odd-symmetric:

$$\phi_{N-n+1} = -\phi_n \quad , \quad n = 1, 2, \ldots, N \; . \tag{25}$$

Since the $\{d_n\}$ in Eq. (23) are odd-symmetric by Eq. (2), for Eq. (25) to hold it is then necessary that the coefficients $\{\gamma_k\}$ be real.

Equation (23) does not provide an explicit solution for the phase perturbations since the coefficients $\{\gamma_k\}$ are defined in terms of the unknown Lagrangian multipliers $\{\mu_k\}$. Nevertheless, the form of the phase perturbations is useful because (a) it makes possible an interpretation of phase-only nulling in terms of cancellation beams; (b) it can serve as the basis of a numerical method for calculating the

23. Shore, R.A. (1983) A Proof of the Odd-Symmetry of the Phases for Minimum Weight Perturbation, Phase-Only Null Synthesis, RADC-TR-83-96.

phase perturbations for null synthesis with the number of unknowns equal to the number of imposed nulls, K, rather than the number of elements in the array, N; and (c) it can be incorporated into adaptive algorithms for phase-only nulling. The application of Eq. (23) to null synthesis is the central concern of this report and is treated in Section 3. The use of Eq. (23) in adaptive nulling is briefly discussed in Section 4. A beam space interpretation of phase-only nulling has been given by Baird and Rassweiler, and also in Reference 13. For completeness, however, we give it briefly here as well, in a slightly more general form.

Assume that the coefficients $\{\gamma_k\}$ are real and that the unperturbed phases, $\{\psi_n\}$, are of the form

$$\psi_n = -d_n u_s \quad , \quad n = 1, 2, \ldots, N \quad ,$$

so that the main beam of the unperturbed pattern is directed towards $u = u_s$. We rewrite Eq. (23) as

$$\phi_n = \text{phase}\left[c_n |a_n| - \sum_{k=1}^{K} \gamma_k \, e^{-jd_n(u_k - u_s)}\right]$$

$$= \tan^{-1}\left\{\frac{\sum_{k=1}^{K} \gamma_k \sin[d_n(u_k - u_s)]}{c_n |a_n| - \sum_{k=1}^{K} \gamma_k \cos[d_n(u_k - u_s)]}\right\} \quad , \quad n = 1, 2, \ldots, N \, , \tag{26}$$

and assume that the phase perturbations are small so that ϕ_n can be approximated by

$$\phi_n \approx \frac{\sum_{k=1}^{K} \gamma_k \sin[d_n(u_k - u_s)]}{c_n |a_n| - \sum_{k=1}^{K} \gamma_k \cos[d_n(u_k - u_s)]} \quad , \quad n = 1, 2, \ldots, N \, , \tag{27}$$

and the weight perturbations,

$$\Delta w_n = a_n \left(e^{j\phi_n} - 1\right) \quad , \quad n = 1, 2, \ldots, N \, ,$$

approximated by

$$\Delta w_n \approx j a_n \phi_n$$

$$\approx \frac{j\, e^{-jd_n u_s} \sum_{k=1}^{K} \gamma_k \sin\left[d_n(u_k - u_s)\right]}{c_n - \frac{1}{|a_n|} \sum_{k=1}^{K} \gamma_k \cos\left[d_n(u_k - u_s)\right]}$$

$$= \frac{1}{2} \frac{\sum_{k=1}^{K} \gamma_k \left[e^{jd_n(u_k - 2u_s)} - e^{-jd_n u_k}\right]}{c_n - \frac{1}{|a_n|} \sum_{k=1}^{K} \gamma_k \cos\left[d_n(u_k - u_s)\right]} , \quad n = 1, 2, \ldots, N . \tag{28}$$

The cancellation pattern

$$\Delta p(u) = \sum_{n=1}^{N} \Delta w_n \, e^{jd_n u}$$

is then approximated by

$$\Delta p(u) \approx \frac{1}{2} \sum_{k=1}^{K} \gamma_k \sum_{n=1}^{N} \frac{1}{c_n'} \left\{ e^{jd_n[u-(2u_s-u_k)]} - e^{jd_n(u-u_k)} \right\} , \tag{29}$$

where

$$c_n' \triangleq c_n - \frac{1}{|a_n|} \sum_{k=1}^{K} \gamma_k \cos\left[d_n(u_k - u_s)\right] , \tag{30}$$

and so is the sum of K pairs of beams, one member of each pair directed towards an imposed null location, $u = u_k$, and the other member, of opposite sign, directed towards the symmetric location with respect to the direction of the unperturbed main beam,

$$u = -u_k + 2u_s = u_k - 2(u_k - u_s) \quad .$$

The shape of the cancellation beams is determined by the $\{c_n'\}$. For $c_n = 1$, $n = 1, 2, \ldots, N$, and the magnitude of the beam coefficients small compared with the $\{|a_n|\}$, the beams are of the form $\sin\left(\frac{Nu}{2}\right)\Big/\sin\left(\frac{u}{2}\right)$; that is, beams corresponding to an array of N elements with uniform amplitude. For other choices of the $\{c_n\}$ the cancellation beams correspond to arrays with tapered amplitude distributions. Further details may be found in References 13 and 15.

Although the interpretation of the phase-only nulling cancellation pattern as a sum of beam pairs is based, as we have seen, on the assumption of small phase perturbations, we shall for convenience in this report refer in general to Eqs. (23) and (26) as "beam space" representations of phase-only weight perturbations, and to the $\{\gamma_k\}$ as "beam coefficients." It should be kept in mind, however, that strictly speaking the beam space interpretation applies only to phase perturbations small enough for the approximations, Eqs. (27) and (28), to be acceptable, say $|\phi_n| < 0.5$ (rad). It should also be noted that the beam space representation, (29), of the cancellation pattern is in general a nonlinear superposition of beams because the beam coefficients, $\{\gamma_k\}$, enter into the expression, Eq. (30), for the $\{c_n'\}$. Only if the beam coefficients are negligibly small compared to the $\{c_n|a_n|\}$ does the representation, (29), become a linear superposition of beams.

3. NUMERICAL CALCULATION OF PHASE-ONLY WEIGHT PERTURBATIONS FOR NULL SYNTHESIS

The beam space representation, (26), obtained in the previous section for the phase perturbations for minimized weight perturbation null synthesis, can be used as the basis of a highly efficient method for numerically calculating the phase perturbations. To calculate the beam coefficients, $\{\gamma_k\}$, in (26) we use computer algorithms that have been developed for solving the so-called nonlinear programming problem — the problem of minimizing or maximizing a nonlinear function of several variables subject to a set of nonlinear equality and/or inequality constraints. Here, the nonlinear function we wish to minimize is given by Eq. (7), which we rewrite in the form

$$F = 2 \sum_{n=1}^{N} c_n |a_n|^2 (1 - \cos \phi_n)$$

$$= 4 \sum_{n=1}^{N} c_n |a_n|^2 \sin^2 \left(\frac{\phi_n}{2}\right) \tag{31}$$

and the nonlinear equality constraints are, from Eq. (3) and the odd-symmetry of the $\{d_n\}$ and the $\{\phi_n\}$,

$$E_k \triangleq \sum_{n=1}^{N} |a_n| \cos [\phi_n + d_n(u_k - u_s)] = 0 \quad , \quad k = 1, 2, \ldots, K \quad . \tag{32}$$

In Reference 18 we investigated the solution of this nonlinear programming problem when the $\{\phi_n\}$ are the unknowns. The performance on this problem of the two nonlinear programming algorithms LPNLP[24] and VMCON[25] was compared. Here we assume the form of (26) for the phase perturbations with the K beam coefficients $\{\gamma_k\}$ as the unknown variables. In applications to null synthesis with large arrays and the number of imposed nulls, K, small relative to the number of independent phases, N/2, this approach has the obvious merit of reducing the number of degrees of freedom from N/2 to K. Since many nonlinear programming algorithms, including LPNLP and VMCON, require explicit expressions for the derivatives of the objective function and of the constraint functions with respect to the unknowns (some algorithms compute these derivatives via discrete difference approximations), we give expressions for the derivatives of F and the $\{E_k\}$ with respect to the $\{\gamma_k\}$ in Appendix A.

To investigate the relative performance of the two approaches, beam space and "phase space," to calculate the phase-only weight perturbations, the beam coefficient method was used to calculate the phases for the same null synthesis problems as were used in the study described in Reference 18. The first problem was that of imposing nulls in the pattern of a 41-element array with uniform amplitude and half-wavelength interelement spacing. The amplitudes of the elements

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24. Pierre, D.A., and Lowe, M.J. (1975) Mathematical Programming Via Augmented Lagrangians, Addison-Wesley, Mass.

25. Crane, R.L. et al (1980) Solution of the General Nonlinear Programming Problems With Subroutine VMCON, Report ANL-80-64, Argonne National Laboratory, Argonne, Ill.

and the weighting coefficients $\{c_n\}$ were set equal to unity, and the direction of the unperturbed main beam, u_s, was taken to be zero (that is, broadside to the array). A series of sets of imposed null locations was used starting with one null at 4.0^o, then two nulls at 4.0^o and 4.6^o, up through five nulls at 4.0^o. 4.6^o, 5.2^o, 5.8^o, and 6.4^o. Both LPNLP and VMCON were run in double precision on a CDC 6600 computer. In running LPNLP the modified Davidon-Fletcher-Powell (DFP) conjugate gradient mode (ISS = 0) was used without automatic reset to the gradient direction (IRESET = 0). The convergence parameters were set at $\epsilon_1 = 1.0 \times 10^{-10}$, $\epsilon_2 = \epsilon_3 = 1.0 \times 10^{-8}$. In running VMCON, the tolerance was set at 1.0×10^{-10}. Both programs were run with the unknown beam coefficients set initially to zero.

In Table 1 we compare null depths and computation times obtained for this test problem with LPNLP and VMCON for beam-space and phase-space nulling. The phase-space nulling results are taken directly from Reference 18. The most striking result is the remarkable improvement in computation time obtained with beam-space nulling as compared with phase-space nulling. This improvement is especially impressive when VMCON was used as the nonlinear programming algorithm; for the one-, two-, and three-null cases, computation time decreased by a factor of more than 50, while for the four-null case the computation time was reduced by a factor of 15. Smaller yet considerable decreases in computation time were obtained when LPNLP was used as the algorithm. Null depths obtained with the two methods are comparable. When convergence to a solution occurred, the beam space method always gave the same phases as the phase space method to within very small differences. The five-null case could not be solved when the beam space method was used.

To help understand the failure of the beam space method in the five-null case, in Table 2 we have tabulated the beam coefficients and the average absolute phase perturbation as a function of the number of imposed nulls. The beam coefficients are tabulated in increasing order of the imposed null locations towards which the respective cancellation beams are directed. The average of the magnitudes of the beam coefficients increases by a factor of more than 70 and the average absolute phase perturbation doubles from the one-null to the four-null case. As discussed in Reference 18, this test problem was intentionally chosen to be a difficult one, requiring nulls to be imposed at closely-spaced locations in a near-in sidelobe region of a uniform array. Viewed intuitively in terms of a picture of cancellation beams, the magnitudes of the beam coefficients increase strongly with the number of nulls in such a nulling situation because the main lobes of the cancellation beams interfere with each other, thus requiring extensive mutual adjustment of the magnitudes of the beams for nulling to occur. This interference is especially evidenced by the four-null case in Table 2 in which not only are

Table 1. Comparison of Beam Space and Phase Space Phase-Only Nulling in the Pattern of a 41-Element, Uniform Amplitude Array With $\lambda/2$ Spacing. Nulls imposed at the series of locations 4.0°, 4.6°, 5.2°, 5.8°, and 6.4°

		Number of Nulls									
		1		2		3		4		5	
		Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)
LPNLP	Beam	-284	0.8	<-239	1.6	<-254	5.7	<-222	64.9	--	--
	Phase	-255	2.9	<-289	5.7	<-265	12.0	<-249	68.9	<-186	1007
VMCON	Beam	-273	0.4	<-272	0.6	<-289	0.7	<-289	2.5	--	--
	Phase	-266	25.4	<-254	32.5	<-275	39.5	<-283	34.4	--	--

Table 2. Values of the Beam Coefficients and the Average Absolute Phase Perturbation for Beam Space Phase-Only Nulling in the Pattern of a 41-Element Uniform Amplitude Array With $\lambda/2$ Spacing. Nulls imposed at the series of locations 4.0°, 4.6°, 5.2°, and 5.8° using LPNLP and VMCON

Number of Nulls	Beam Coefficients	Average Absolute Phase Perturbation (rad)
1	-0.4	0.28
2	-0.9 +0.5	0.31
3	+1.8 -4.3 +2.6	0.33
4	+15.6 -42.9 +42.0 -14.9	0.54

the magnitudes of the beams all much larger than that required to impose a single null at the respective locations,* but in which the beam coefficients alternate in sign despite the fact that the original pattern is negative at the locations 4.0°, 4.6°, and 5.2°, and is positive only at 5.8°. The increase in the magnitude of the beam coefficients with the number of closely-spaced nulls is clearly associated with an increase in numerical difficulty as shown by the increase in computation time, especially pronounced in going from the three- to the four-null case. The numerical difficulty can be attributed in part to the fact that increasingly large changes in the beam coefficients are required to bring about small changes in the objective and constraint functions. However, even when the phases are taken to be the unknowns, the five-null cases in this series of imposed nulls presents formidable difficulty as can be seen in Table 1 from the fact that only LPNLP converged to a solution and required an inordinately large amount of computation time. Further work is needed to develop methods for handling such difficult phase-only nulling situations.

The second test problem used to compare the beam-space and phase-space nulling methods was that of imposing nulls in the same pattern of an array with 41 elements, half-wavelength spacing, and uniform amplitude, at the series of

*If a single null is imposed at each of the locations 4.0°, 4.6°, 5.2°, and 5.8°, the value of the beam coefficient is respectively -0.41, -0.31, -0.14, and +0.07.

sets of locations 25°; 25° and 35°; and so on up through a set of five imposed nulls at 25°, 35°, 45°, 55°, and 65°. This test problem, in contrast to the first problem, was chosen to be relatively easy, requiring nulls to be imposed at wide-spaced locations in the lower sidelobe region of the pattern. In Tables 3 and 4 we have tabulated the same quantities for this test problem as in Tables 1 and 2, respectively. Again the phase space results are taken directly from Reference 18. As was noted for Table 1, the salient feature of Table 3 is the remarkable reduction in computation time made possible by employing the beam-space nulling method, especially when VMCON is used as the nonlinear programming algorithm. Even the five-null case was solved by VMCON using the beam space method in about half the shortest time required to solve the one-null case when the phase space method was used. Examining the beam coefficients in Table 4, it is seen that there is relatively little change in their values as nulls are added; the coefficient associated with the 25° beam varies between 0.067 and 0.065, the beam coefficient associated with the 35° beam varies between -0.039 and -0.040, and so on. Again viewed intuitively in terms of a picture of cancellation beams, in this test example the beams are coupled only through their sidelobes instead of through their main beams as in the first test problem. Hence, the cancellation beams are much more independent of each other and the beam coefficients for the multiple imposed null cases are very close to the values they have when only a single null is imposed at each of the respective locations (+0.067, -0.042, +0.055, +0.029, and +0.046 for a single imposed null at 25°, 35°, 45°, 55°, and 65°, respectively). Because of the relative independence of the cancellation beams, little mutual adjustment of the beams is required for nulling and the numerical difficulty of the problem increases only slightly as additional nulls are imposed.

It may be of interest to note that the values of the beam coefficients for the second test problem can be estimated rather closely from a simplified form of the approximation, (29), for the cancellation pattern. In this example the weighting coefficients $\{c_n\}$ and the amplitudes of the element weights $\{|a_n|\}$ are equal to unity, the unperturbed main beam direction, u_s, equals zero, and N, the number of array elements, is 41. Assuming that the $\{|\gamma_k|\}$ are small relative to one, we can approximate the $\{c_n'\}$ in Eq. (30) by $c_n = 1$. At the imposed null locations $u = u_k$, $k = 1, 2, \ldots, K$, the cancellation pattern, $\Delta p(u)$, can then be approximated by

$$\Delta p(u_k) \approx -\frac{N}{2}\gamma_k \quad , \quad k = 1, 2, \ldots, K \quad ,$$

if we neglect the contributions to the cancellation pattern at $u = u_k$ of the beams directed towards the other imposed null locations and towards the locations

Table 3. Comparison of Beam Space and Phase Space Phase-Only Nulling in the Pattern of a 41-Element, Uniform Amplitude Array With $\lambda/2$ Spacing. Nulls imposed at the series of locations 25^{o}, 35^{o}, 45^{o}, 55^{o}, and 65^{o}

		Number of Nulls									
		1		2		3		4		5	
		Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)
LPNLP	Beam	-282	0.6	<-277	1.1	<-276	1.9	<-277	3.1	<-274	4.7
	Phase	-276	2.3	<-275	3.6	<-281	4.7	<-277	5.8	<-282	7.1
VMCON	Beam	-282	0.4	<-277	0.5	<-280	0.8	<-277	1.1	<-274	1.4
	Phase	-283	12.1	<-280	12.1	<-239	15.0	<-240	14.5	<-241	16.0

Table 4. Values of the Beam Coefficients and the Average Absolute Phase Perturbation for Beam Space Phase-Only Nulling in the Pattern of a 41-Element Uniform Amplitude Array With $\lambda/2$ Spacing. Nulls imposed at the series of locations 25^o, 35^o, 45^o, 55^o, and 65^o using LPNLP and VMCON

Number of Nulls	Beam Coefficients	Average Absolute Phase Perturbation (rad)
1	+0.067	0.044
2	+0.064 -0.039	0.044
3	+0.066 -0.040 +0.054	0.053
4	+0.065 -0.040 +0.052 +0.018	0.054
5	+0.065 -0.039 +0.052 +0.020 +0.046	0.063

symmetric with respect to the main beam of the imposed null locations. Since by definition the cancellation pattern equals the negative of the value of the unperturbed pattern at the imposed null locations,

$$\Delta p(u_k) = -p_o(u_k) \quad , \quad k = 1, 2, \ldots, K \quad , \tag{33}$$

it follows that

$$\gamma_k = \frac{2}{N} p_o(u_k) \quad , \quad k = 1, 2, \ldots, K \quad . \tag{34}$$

Since the value of the unperturbed pattern at the locations 25^o, 35^o, 45^o, 55^o, and 65^o is 1.41, -0.88, 1.12, 0.63, and 0.98, respectively, we obtain from the approximation, (34), as estimates of the beam coefficients 0.069, -0.043, 0.054, 0.031, and 0.048. These values are quite close to the calculated values given in Table 4. More accurate estimates can be obtained from the approximation, (29), by replacing the $\{c'_n\}$ by c_n as above but not neglecting the contributions of any of the cancellation beams. Equation (33) then yields a system of K simultaneous linear equations that can be solved for the beam coefficients.

4. ADAPTIVE PHASE-ONLY NULLING

The application to adaptive phase-only nulling of the beam space representation of the phase perturbations has been discussed in detail by Baird and Rassweiler,[1] and we will here touch only briefly on this topic.

In the previous section we showed that computation efficiency for minimized weight perturbation, phase-only null synthesis in purely real patterns could be significantly increased by using a beam space representation of the phase perturbations, thereby reducing the number of unknowns from N/2 (N the number of elements) to K, the number of imposed nulls. The beam coefficients were determined using nonlinear programming algorithms supplied with values of the objective function, the constraint functions, and their derivatives with respect to the beam coefficients. In adaptive nulling with realistic arrays, the pattern can no longer be assumed real and, consequently, the beam coefficients $\{\gamma_k\}$ in the beam space representation, Eq. (23), must be regarded as complex constants. Equation (23) can be used to reduce the number of unknowns from N to 2K provided that the directions $\{u_k\}$ of the sources of interference are known with sufficient accuracy.

In adaptive nulling it is also often the case that the only information available to the adaptive algorithm (other than that of the directions of signal and jammers) is that of the total output power of the array. The beam coefficients must then be determined by random search or gradient methods that attempt to reduce the total output power to a minimum. One danger of such algorithms is that the desired signal will be nulled out in the course of reducing the total output power. Constrained gradient methods such as that developed by Frost[26] are not helpful in phase-only adaptive nulling since they require adjustments of the element amplitudes as well as the phases. One distinct advantage of using the beam space representation for phase-only adaptive nulling, as compared with adapting with the phases themselves as the control variables, is that in certain situations (interference sources spaced not too closely in a low sidelobe region of an antenna pattern) the beam coefficients are small in magnitude and the cancellation pattern then consists of beams with complex coefficients directed at the interference sources and at the locations symmetric with respect to the main beam. These cancellation beams can have only a small effect on the main beam and so the

26. Frost, O. L. (1971) An algorithm for linearly constrained adaptive array processing, Proc. IEEE 60:661-675.

response of the antenna to the desired signal is not likely to be significantly degraded as a result of the adaptive adjustment of the phases. The choice of the weighting coefficients $\{c_n\}$ in Eq. (23) can be used to control the shape of the cancellation beams. If $c_n = 1$ for all elements, the cancellation beams are of the form $\sin\left(\frac{Nu}{2}\right)/\sin\left(\frac{u}{2}\right)$ corresponding to a uniform amplitude array. Other choices of the $\{c_n\}$ can be used to obtain cancellation beams with broader main beams and lower sidelobes. These may be desirable in certain applications. Further details may be found in Reference 13. For closely-spaced jammers the beam space method is likely to present considerable difficulties in adaptive nulling because the beam coefficients are then large, and instabilities can easily arise from the kind of phenomena noted in the previous section in regard to the first test problem.

5. CONCLUSIONS

In this report we have described a new computational method for minimized weight perturbation, phase-only null synthesis in linear array antenna patterns, consisting of the use of nonlinear programming computer algorithms to calculate the coefficients of a beam space representation of the optimal phase perturbations. The beam space representation enables the number of unknowns to be reduced from half the number of array elements to the number of imposed null locations. The derivatives of the objective and null constraint functions with respect to the beam coefficients, utilized by the nonlinear programming algorithms, are calculated from analytic expressions. Computation time is significantly reduced in general by this method, compared with that required to calculate the phase perturbations directly with nonlinear programming techniques.

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Appendix A

Calculation of the Derivatives of the Objective Function and Constraint Functions With Respect to Beam Coefficients

In this appendix we obtain expressions for the derivatives with respect to the beam coefficients $\{\gamma_k\}$ of the objective function F given by Eq. (31) and the constraint functions $\{E_k\}$ given by Eq. (32).

From Eq. (31)

$$\frac{\partial F}{\partial \gamma_p} = 4 \sum_{n=1}^{N} c_n |a_n|^2 \left[2 \sin\left(\frac{\phi_n}{2}\right) \cos\left(\frac{\phi_n}{2}\right) \frac{1}{2} \right] \frac{\partial \phi_n}{\partial \gamma_p}$$

$$= 2 \sum_{n=1}^{N} c_n |a_n|^2 \sin \phi_n \frac{\partial \phi_n}{\partial \gamma_p} \ , \quad p = 1, 2, \ldots, K \ , \tag{A1}$$

and from Eq. (32)

$$\frac{\partial E_k}{\partial \gamma_p} = \sum_{n=1}^{N} \frac{\partial E_k}{\partial \phi_n} \frac{\partial \phi_n}{\partial \gamma_p}$$

$$= - \sum_{n=1}^{N} |a_n| \sin [\phi_n + d_n(u_k - u_s)] \frac{\partial \phi_n}{\partial \gamma_p} , \quad p = 1, 2, \ldots, K . \tag{A2}$$

To calculate $\frac{\partial \phi_n}{\partial \gamma_p}$ from Eq. (26), we let

$$y_n = \frac{\sum_{k=1}^{K} \gamma_k \sin [d_n(u_k - u_s)]}{c_n |a_n| - \sum_{k=1}^{K} \gamma_k \cos [d_n(u_k - u_s)]} , \tag{A3}$$

so that

$$\phi_n = \tan^{-1} y_n \tag{A4}$$

and

$$y_n = \tan \phi_n . \tag{A5}$$

Then,

$$\frac{\partial \phi_n}{\partial \gamma_p} = \frac{d\phi_n}{dy_n} \frac{\partial y_n}{\partial \gamma_p} . \tag{A6}$$

From Eqs. (A4) and (A5)

$$\frac{d\phi_n}{dy_n} = \frac{1}{1 + y_n^2} = \frac{1}{1 + \tan^2 \phi_n} = \cos^2 \phi_n \tag{A7}$$

and from Eq. (A3)

$$\frac{\partial y_n}{\partial \gamma_p} = \frac{1}{v_n} \sin [d_n(u_p - u_s)] + \frac{t_n}{v_n^2} \cos [d_n(u_p - u_s)] \tag{A8}$$

where

$$v_n \triangleq c_n |a_n| - \sum_{k=1}^{K} \gamma_k \cos [d_n(u_k - u_s)]$$

and

$$t_n \triangleq \sum_{k=1}^{K} \gamma_k \sin [d_n(u_k - u_s)] \quad .$$

Substituting Eqs. (A4) and (A8) in Eq. (A6) we obtain

$$\frac{\partial\phi_n}{\partial\gamma_p} = \frac{\cos^2 \phi_n}{v_n} \left\{ \sin [d_n(u_p - u_s)] + \frac{t_n}{v_n} \cos [d_n(u_p - u_s)] \right\} \quad . \tag{A9}$$

The derivatives of the objective function and constraint functions are then given, respectively, by Eqs. (A1) and (A2) with $\frac{\partial\phi_n}{\partial\gamma_p}$ obtained from Eq. (A9).

Appendix B

Comparison of the Minimized Weight Perturbation Objective Function With the Objective Function Used by Baird and Rassweiler

As noted in the introduction, Baird and Rassweiler[B1] treat the problem of minimizing, by phase-only control, the mean square error between a desired signal and the array output, given discrete sinusoidal sources of signal and interferences. In the special case that the interferences are infinite in power, the mean square error can be minimized only by placing pattern nulls in the direction of the interference. The mean square difference between desired signal and array output is then simply $|p(u_s) - p_o(u_s)|^2$, where $p_o(u_s)$ and $p(u_s)$ are the values of the original and perturbed pattern, respectively, at $u = u_s$, the direction of the desired signal. For the main beam of the original pattern to be directed towards $u = u_s$, the unperturbed weights are of the form $a_n = |a_n|\, e^{-jd_n u_s}$ and

$$|p(u_s) - p_o(u_s)|^2 = \left| \sum_{n=1}^{N} a_n e^{j\phi_n} e^{jd_n u_s} - \sum_{n=1}^{N} a_n e^{jd_n u_s} \right|^2$$

$$= \left| \sum_{n=1}^{N} \left[|a_n| \left(e^{j\phi_n} - 1 \right) \right] \right|^2 \quad . \tag{B1}$$

B1. Baird, C.A., and Rassweiler, G.G. (1976) Adaptive sidelobe nulling using digitally controlled phase-shifters, IEEE Trans. Antennas Propag. AP-24:638-649.

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where the $\{\phi_n\}$ are the phase perturbations. This objective function differs in general from the minimized weight perturbation objective function we employ [Eq. (4)],

$$F = \sum_{n=1}^{N} c_n |a_n|^2 \left| e^{j\phi_n} - 1 \right|^2$$

$$= 2 \sum_{n=1}^{N} c_n |a_n|^2 (1 - \cos \phi_n) \quad , \qquad \text{(B2)}$$

even when $c_n = |a_n| = 1$, $n = 1, \ldots, N$, the case treated by Baird and Rassweiler. Only if the additional restriction of the phase perturbations to be odd-symmetric is made do the two objective functions become equivalent, since then

$$\left| \sum_{n=1}^{N} \left(e^{j\phi_n} - 1 \right) \right|^2 = \left[\sum_{n=1}^{N} (\cos \phi_n - 1) \right]^2$$

$$= \left(\frac{F}{2} \right)^2 \quad .$$

Even though the limiting form, for infinite power interferences, of the objective function employed by Baird and Rassweiler, Eq. (B1), differs from the minimized weight perturbation objective function, Eq. (B2), the two objective functions lead to the same form of beam space representation for the phase perturbations when $c_n = |a_n| = 1$, as we have noted in Section 2 regarding Eq. (24). To understand why this is so, we briefly trace through the derivation of Eq. (24), using Eq. (B1) as the objective function. Starting with Eq. (10), the equation requiring the Lagrangian to have a zero gradient, it is seen that if the objective function, Eq. (B1), is used instead of F, all terms on the right-hand side of Eq. (10) remain the same except for $\nabla_{\underline{w}} F$, which is replaced by $\nabla_{\underline{w}} G$, where

$$G \triangleq \left| \sum_{n=1}^{N} \left[|a_n| \left(e^{j\phi_n} - 1 \right) \right] \right|^2$$

$$= \sum_{m=1}^{N} \sum_{n=1}^{N} |a_m| \, |a_n| (w_m^* - 1)(w_n - 1)$$

using Eq. (5). To calculate $\nabla_{\underline{w}} G$ we let

$$\underline{w}' = \underline{w} - \underline{1} \quad ,$$

where

$$\underline{1} = [1, 1, \ldots 1]^T \quad ,$$

so that

$$G = \underline{w}'^{\dagger} \underline{P} \, \underline{w}' \quad ,$$

where $\underline{P}$ is the symmetric matrix

$$\underline{P} = \underline{\beta} \, \underline{\beta}^T$$

with

$$\underline{\beta} = \Big[|a_1| \, , \, |a_2| \, , \, \ldots , \, |a_N| \Big]^T \quad .$$

Then

$$\begin{aligned} \nabla_{\underline{w}} G &= \nabla_{\underline{w}'} G \\ &= 2 \, \underline{P} \, \underline{w}' \\ &= 2 \, \underline{P} \, (\underline{w} - \underline{1}) \quad , \end{aligned}$$

using the formula for the complex gradient of a Hermitian quadratic form as in obtaining Eq. (14). Equation (17) is then replaced by

$$\underline{w} = (\underline{P} + \underline{R} + \underline{\Lambda})^{-1} (\underline{P} \, \underline{1}) \quad .$$

The recursive matrix inversion procedure used to obtain Eq. (20) now gives

$$\underline{w} = \underline{\Lambda}^{-1} \left(\underline{P}\,\underline{1} - \sum_{k=1}^{K} \gamma'_k \underline{v}_k - \gamma'_o \underline{\beta} \right) ,$$

where the $\{\underline{v}_k\}$ are given by Eq. (13) and the $\{\gamma'_k\}$ are scalar coefficients. Since

$$\begin{aligned} \underline{P}\,\underline{1} - \gamma'_o \underline{\beta} &= \underline{\beta}\,\underline{\beta}^T \underline{1} - \gamma'_o \underline{\beta} \\ &= (\underline{\beta}^T \underline{1} - \gamma'_o)\, \underline{\beta} \ , \end{aligned}$$

it follows that

$$\underline{w} = (\underline{\beta}^T \underline{1} - \gamma'_o)\, \underline{\Lambda}^{-1} \left(\underline{\beta} - \sum_{k=1}^{K} \gamma''_k \underline{v}_k \right) ,$$

where

$$\gamma''_k = \frac{\gamma'_k}{\underline{\beta}^T \underline{1} - \gamma'_o} \quad , \quad k = 1, 2, \ldots, K \ .$$

Hence, just as Eq. (23) is obtained from Eq. (20), the phase perturbations are given by

$$\begin{aligned} \phi_n &= \text{phase} \left(|a_n| - e^{jd_n u_s} |a_n| \sum_{k=1}^{K} \gamma''_k e^{-jd_n u_k} \right) \\ &= \text{phase} \left(1 - e^{jd_n u_s} \sum_{k=1}^{K} \gamma''_k e^{-jd_n u_k} \right) \quad , \quad n = 1, 2, \ldots, N , \end{aligned}$$

and the total phases of the element weights by

$$\phi_n - d_n u_s = \text{phase} \left(e^{-jd_n u_s} - \sum_{k=1}^{K} \gamma''_k e^{-jd_n u_k} \right) \quad , \quad n = 1, 2, \ldots, N \ . \qquad \text{(B3)}$$

Note that the amplitudes of the weights do not appear explicitly in these expressions for the phase perturbations and total phases as they do in Eqs. (23) and (24). The values of the coefficients $\{\gamma''\}$ do, however, depend on the weight amplitudes. Comparing Eq. (B3) with Eq. (24), equating ϕ_n with $-d_n u_s$, it is seen that for $c_n = |a_n| = 1$, the two solutions are identical in form. Thus, the minimized signal perturbation objective function, Eq. (B1), and the minimized weight perturbation objective function, Eq. (B2), lead to the same form of representation for the phase perturbations when $c_n = |a_n| = 1$.